2882. [2003 : 466] *Proposed by Mihály Bencze, Brasov, Romania.*

If $x \in \left(0, \frac{\pi}{2}\right)$, $0 \le a \le b$, and $0 \le c \le 1$, prove that

$$\left(\frac{c+\cos x}{c+1}\right)^b \;<\; \left(\frac{\sin x}{x}\right)^a .$$

Solution by Michel Bataille, Rouen, France.

We add the hypothesis $(a,b) \ne (0,0)$, since the inequality is false for $a = b = 0$. If $a = 0$, then $b > 0$, and the function $f(x) = t^b$ is strictly increasing on $(0, \infty)$. Since $0 < \dfrac{c+\cos x}{c+1} < 1$, we have

$$\left(\frac{c+\cos x}{c+1}\right)^b \;<\; 1 \;=\; \left(\frac{\sin x}{x}\right)^0 .$$

Suppose now that $0 < a \le b$. Letting $r = \dfrac{b}{a}$, the proposed inequality becomes $\left(\dfrac{c+\cos x}{c+1}\right)^r < \dfrac{\sin x}{x}$. Since $r \ge 1$, we see that

$$\left(\frac{c+\cos x}{c+1}\right)^r \;\le\; \frac{c+\cos x}{c+1} .$$

Hence, it suffices to prove that $\dfrac{c+\cos x}{c+1} < \dfrac{\sin x}{x}$. But for a fixed $\alpha \in (0,1)$, the function $f(t) = \dfrac{t+\alpha}{t+1} = 1 - \dfrac{1-\alpha}{t+1}$ is clearly increasing on $[0,1]$. Thus, it suffices to show that $\dfrac{1+\cos x}{2} < \dfrac{\sin x}{x}$, which is equivalent to

$$x\cos^2\left(\frac{x}{2}\right) \;<\; 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right), \qquad \text{or} \qquad \frac{x}{2} \;<\; \tan\left(\frac{x}{2}\right) .$$

The last inequality is well known to be true for $x \in (0, \pi)$ and our proof is complete.

Also solved by CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Besides Bataille, four other solvers noticed and stated that the proposed inequality is not valid without additional constraints on a and b. Guersenzvaig assumed that $b \ne 0$. Hess and Janous excluded the case $a = b = 0$. Zhou assumed that $a > 0$ or $a < b$. It is easy to see that all these hypotheses are equivalent to that used by Bataille.

2883. [2003 : 466] *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z \in [0,1)$ and that $x + y + z = 1$. Prove that

$$\sqrt{\frac{xy}{z+xy}} + \sqrt{\frac{yz}{x+yz}} + \sqrt{\frac{zx}{y+zx}} \;\le\; \frac{3}{2} .$$

Essentially the same solution by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Vasile Cîrtoaje, University of Ploiesti, Romania; Titu Zvonaru, Bucharest, Romania; and the proposers.

Since $z + xy = z(x+y+z) + xy = (x+z)(y+z)$, the AM–GM Inequality yields

$$\sqrt{\frac{xy}{z+xy}} = \sqrt{\frac{xy}{(x+z)(y+z)}} \leq \frac{1}{2}\left(\frac{x}{x+z} + \frac{y}{y+z}\right).$$

Hence,

$$\begin{aligned}\sum_{\text{cyclic}} \sqrt{\frac{xy}{z+xy}} &\leq \frac{1}{2}\sum_{\text{cyclic}}\left(\frac{x}{x+z} + \frac{y}{y+z}\right)\\ &= \frac{1}{2}\left(\frac{x}{x+z} + \frac{y}{y+z} + \frac{y}{y+x} + \frac{z}{z+x} + \frac{z}{z+y} + \frac{x}{x+y}\right)\\ &= \frac{1}{2}\left(\frac{x+y}{x+y} + \frac{y+z}{y+z} + \frac{z+x}{z+x}\right) = \frac{3}{2}.\end{aligned}$$

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; VASILE CÎRTOAJE, University of Ploiesti, Romania (a second solution); CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA (two solutions); ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers (a second solution).

Though it is easy to show that the equality holds if and only if $x = y = z = \frac{1}{3}$, only Bornsztein, Specht, Woo, Zvonaru, and the proposer explicitly mentioned this, with Zvonaru being the only one who actually gave a full proof.

Zhao commented that if we replace x, y, z with $\frac{1}{bc}$, $\frac{1}{ca}$, and $\frac{1}{ab}$, respectively (assuming $x, y, z > 0$ since the inequality is trivial if any of x, y, z is zero), then the constraint becomes $a+b+c = abc$ and the inequality becomes

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2},$$

which is well known and appeared in the 1998 Korean Math Olympiad.

2884. [2003 : 467] *Proposed by Niels Bejlegaard, Copenhagen, Denmark.*

Suppose that a, b, c are the sides of a non-obtuse triangle. Give a geometric proof and hence, a geometric interpretation of the inequality

$$a+b+c \geq \sum_{\text{cyclic}} \sqrt{a^2+b^2-c^2}.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Lemma. Suppose that A and B are two points outside a circle centred at O such that AB intersects the circle. If X and Y are two points on the circle such that AX and BY are tangents, then $AB \geq AX + BY$.